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Quantum-like approach to the transversal and longitudinal beam dynamics. The halo problem

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Abstract. An interpretation of the formation of halo in accelerators based on quantum-like theory by a diffraction model is given in terms of the transversal beam motion. Physical implications of the longitudinal dynamics are also examined.

1 Introduction

Recently the description of the dynamical evolution of high density beams by using the collective models, has become more and more popular. A way of developing this point of view is the quantum-like approach [1] where one considers a time-dependent Schrödinger equation, in both the usual linear and the less usual nonlinear form, as a fluid equation for the whole beam. In this case the squared modulus of the wave function (named beam wave function) gives the distribution function of the particles in space at a certain time [2]. The Schrödinger equation may be taken in one or more spatial dimensions according to the particular physical problem; furthermore the motion of the particles in the configuration space can be considered as a Madelung fluid if one chooses the equation in its linear version.

Although the validity of the model relies only on experiments and on the new predictions which must be verified experimentally, we like to invoke here a theoretical argument that could justify the Schrödinger quantum-like approach. Let us think of particles in motion within a bunch in such a way that the single particle moves under an average force field due to the presence of all others and collides with the neighbouring ones in a complicated manner. It is obviously impossible to follow and describe all the forces deterministically. One then faces a situation where the classical motion determined by the force-field is perturbed continuously by a random term, and one finds immediately a connection with a stochastic process. If one assumes that the process is Markovian and Brownian, one easily arrives at a modification of the equations of motion in such a manner that would be synthesized by a linear Schrödinger equation depending on a physical parameter that has the dimension of action [3,4]. Wave quantum mechanics follows if this parameter coincides with the Planck's constant $\hbar$, whereas the quantum-like theory of beams is obtained if one chooses it as the normalized emittance ϵ [1]. In both cases, the evolution of the system is expressed in terms of a continuous field ψ which defines the so-called Madelung fluid. We may notice that the normalized emittance ϵ with the dimension of an action is the natural choice in the quantum-like theory, that finds the analogue in the Planck's constant $\hbar$ because it reproduces the corresponding area in the phase-space of the particle.

We here point out that, after linearizing the Schrödinger-like equation, for beams in an accelerator, one can use the whole apparatus of quantum mechanics, keeping in mind a new interpretation of the basic parameters (for instance the Planck's constant $\hbar \longrightarrow \epsilon$ where ϵ is the normalized beam emittance). In particular one introduces the propagator $K\left(x_f, t_f | x_i, t_i\right)$ of the Feynman theory for both longitudinal and transversal motion. A procedure of this sort seems effective for a global description of several phenomena such as intrabeam scattering, space-charge, particle focusing, that cannot be treated easily in detail by "classical mechanics". One consequence of this procedure is to obtain information on the creation of the *Halo* around the main beam line by the losses of particles due to the transversal collective motion.

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2 Transversal motion

Let us indeed consider the Schrödinger like equation for the beam wave function

$$\mathrm{i}\epsilon\partial_t\psi = -\frac{\epsilon^2}{2m}\partial_x^2\psi + U(x,t)\,\psi \tag{1}$$

in the linearized case $U(x,t)$ does not depend on the density $|\psi|^2$. ϵ here is the normalized transversal beam emittance defined as follows:

$$\epsilon = m_0 c\gamma\beta\tilde{\epsilon}\,, \tag{2}$$

$\tilde{\epsilon}$ being the emittance usually considered, (we may also introduce the analogue of the de Broglie wavelength as $\lambda = \epsilon/p$). Let us now focus our attention on the one dimensional transversal motion along the x-axis of the beam particles belonging to a single bunch and assume a Gaussian transversal profile for particles injected into a circular machine. We want to try a description of interactions that cannot be treated in detail, as a diffraction through a slit that becomes a phenomenological boundary in each segment of the particle trajectory. This condition should be applied to both beam wave function and beam propagator K. The result is a multiple integral that determines the actual propagator between the initial and final states in terms of the space-time intervals due to the intermediate segments.

$$\begin{aligned}&K\left(x+x_0, T+\tau|x',0\right)\\ &= \int_{-b}^{+b} K\left(x+x_0,\tau|x_0+y_n, T+(n-1)\tau'\right)\\ &\times K\left(x+y_n, T+(n-1)\tau'|x_0+y_{n-1}, T+(n-2)\tau'\right)\\ &\cdots K\left(x+y_1, T|x',0\right)dy_1 dy_2\cdots dy_n\end{aligned} \tag{3}$$

where $\tau = n\tau'$ is the total time spent by the beam in the accelerator (total time of revolutions in circular machines), T is the time necessary to insert the bunch (practically the time between two successive bunches) and $(-b,+b)$ the space interval defining the boundary mentioned above. Obviously b and T are phenomenological parameters which vary from a machine to another and must also have a strict correction with the geometry of the vacuum tube where the particles circulate.

We may consider the two simplest possible approximations for $K(n|n-1) \equiv K(x_0+y_n, T+(n-1)\tau'|x_0+y_{n-1}+(n-2)\tau')$:

1. We substitute the correct K with the free particle K_0 assuming that in the τ' interval ($\tau' \ll \tau$) the motion is practically a free particle motion between the boundaries $(-b,+b)$.
2. We substitute it with the harmonic oscillator $K_\omega\left(n|n-1\right)$ considering the betatron and the synchrotron oscillations with frequency $\omega/2\pi$

3 Free particle case

We may notice that the convolution property (3) of the Feynman propagator allows us to substitute the multiple integral (that becomes a functional integral for $n \longrightarrow \infty$ and $\tau' \longrightarrow 0$) with the single integral

$$\begin{aligned}K(x+x_0, T+\tau|x',0) &= \int_{-b}^{+b} dy K(x+x_0, T+\tau|x_0+y, T)\\ &\times K(x_0+y, T|x',0)dy\end{aligned} \tag{4}$$

After introducing the Gaussian slit $\exp\left[-\frac{y^2}{2b^2}\right]$ instead of the segment $(-b,+b)$ we have

$$\begin{aligned}&K\left(x+x_0, T+\tau|x',0\right)\\ &= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2}\right]\left\{\frac{2\pi\mathrm{i}\hbar\tau}{m}\frac{2\pi\mathrm{i}\hbar T}{m}\right\}^{-\frac{1}{2}}\\ &\quad\times\exp\left[\frac{\mathrm{i}m}{2\hbar\tau}(x-y)^2\right]\exp\left[\frac{\mathrm{i}m}{2\hbar T}(x_0+y-x')^2\right]\\ &= \sqrt{\frac{m}{2\pi\mathrm{i}\hbar}}\left(T+\tau+T\tau\frac{\mathrm{i}\hbar}{mb^2}\right)^{-\frac{1}{2}}\\ &\quad\times\exp\left[\frac{\mathrm{i}m}{2\hbar}\left(v_0^2 T+\frac{x^2}{\tau}\right)+\frac{\left(m^2/2\hbar^2\tau^2\right)\left(x-v_0\tau\right)^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right]\end{aligned} \tag{5}$$

where $v_0 = \frac{x_0-x'}{T}$ and $x_0$is the initial central point of the beam at injection and can be chosen as the origin ($x_0 = 0$) of the transverse motion of the reference trajectory in the frame of the particle. $\hbar$ **must be interpreted as the normalized beam emittance in the quantum-like approach**.

With an initial Gaussian profile (at $t = 0$), the beam wave function (normalized to 1) is

$$f(x) = \left\{\frac{\alpha}{\pi}\right\}^{\frac{1}{4}}\exp\left[-\frac{\alpha}{2}x'^2\right] \tag{6}$$

$\sqrt{\frac{1}{\alpha}}$ being the r.m.s transversal spot size of the beam; the final beam wave function is:

$$\begin{aligned}\phi(x) &= \int_{-\infty}^{+\infty} dx'\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{\left[-\frac{\alpha}{2}x'^2\right]} K\left(x, T+\tau; x',0\right)\\ &= B\exp\left[Cx^2\right]\end{aligned} \tag{7}$$

with

$$\begin{aligned}B &= \sqrt{\frac{m}{2\pi\mathrm{i}\hbar}}\left\{T+\tau+T\tau\frac{\mathrm{i}\hbar}{mb^2}\right\}^{-\frac{1}{2}}\left\{\frac{\alpha}{\pi}\right\}^{\frac{1}{4}}\\ &\times\sqrt{\frac{\pi}{\left(\frac{\alpha}{2}-\frac{\mathrm{i}m}{2\hbar T}-\frac{m^2/2\hbar^2T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right)}}\\ C &= \frac{\mathrm{i}m}{2\hbar\tau}+\frac{m^2/2\hbar^2T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\\ &+\frac{\frac{\tau^2}{T^2}\left\{\frac{m^2/2\hbar^2T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right\}^2}{\left(\frac{\alpha}{2}-\frac{\mathrm{i}m}{2\hbar T}-\frac{m^2/2\hbar^2T^2}{\frac{\mathrm{i}m}{\hbar}\left(\frac{1}{T}+\frac{1}{\tau}\right)-\frac{1}{b^2}}\right)}\end{aligned} \tag{8}$$

The final local distribution of the beam that undergoes the diffraction is therefore

$$\rho(x) = |\phi(x)|^2 = BB^* \exp\left[-\tilde{\alpha}x^2\right] \tag{9}$$

where $\tilde{\alpha} = -(C+C^*)$ and the total probability per particle is given by

$$P = \int_{-\infty}^{+\infty} dx \rho(x) = BB^* \sqrt{\frac{\pi}{\tilde{\alpha}}} \tag{10}$$

Under certain physical conditions (such as the LHC transversal, Table–I), $P \approx \frac{1}{\sqrt{\alpha}} \frac{mb}{\hbar T}$.

4 Oscillator case

Similarly we may consider the harmonic oscillator case (betatronic oscillations and synchrotronic oscillations) to compute the diffraction probability of the single particle from the beam wave function and evaluate the probability of beam losses per particle. The propagator $K_\omega(x, T+\tau|y, T)$ in the later case is:

$$\begin{aligned}
& K(x, T+\tau|x', 0) \\
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2}\right] K_\omega(x, T+\tau|y, T)\, K_\omega(y, T|x', 0) \\
&= \int_{-\infty}^{+\infty} dy \exp\left[-\frac{y^2}{2b^2}\right] \left\{\frac{m\omega}{2\pi \mathrm{i}\hbar \sin(\omega\tau)}\right\}^{\frac{1}{2}} \\
&\quad \times \exp\left[\frac{\mathrm{i}m\omega}{2\hbar \sin(\omega\tau)} \left\{\left(x^2+y^2\right)\cos\omega\tau - 2xy\right\}\right] \\
&\quad \times \left\{\frac{m\omega}{2\pi \mathrm{i}\hbar \sin(\omega T)}\right\}^{\frac{1}{2}} \\
&\quad \times \exp\left[\frac{\mathrm{i}m\omega}{2\hbar \sin(\omega T)} \left\{\left(y^2+x'^2\right)\cos\omega T - 2x'y\right\}\right] \\
&= \left\{\frac{1}{2\pi}\tilde{C}\right\}^{\frac{1}{2}} \exp\left[\tilde{A}x^2 + \tilde{B}x'^2 + \tilde{C}xx'\right]
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\tilde{A} &= \mathrm{i}\frac{m\omega}{2\hbar}\frac{\cos(\omega\tau)}{\sin(\omega\tau)} - \left(\frac{m\omega}{2\hbar}\right)^2 \frac{1}{\sin^2(\omega\tau)}\frac{1}{D}, \\
\tilde{B} &= \mathrm{i}\frac{m\omega}{2\hbar}\frac{\cos(\omega T)}{\sin(\omega T)} - \left(\frac{m\omega}{2\hbar}\right)^2 \frac{1}{\sin^2(\omega T)}\frac{1}{D} \\
\tilde{C} &= -\left(\frac{m\omega}{2\hbar}\right)^2 \frac{2}{\sin(\omega\tau)\sin(\omega T)}\frac{1}{D}, \\
D &= \frac{1}{2b^2} - \mathrm{i}\frac{m\omega}{2\hbar}\left(\frac{\cos(\omega\tau)}{\sin(\omega\tau)} + \frac{\cos(\omega T)}{\sin(\omega T)}\right)
\end{aligned} \tag{12}$$

$$\begin{aligned}
\phi_\omega(x) &= \int_{-\infty}^{+\infty} dx' \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{\alpha}{2}x'^2\right] K_\omega(x, T+\tau; x', 0) \\
&= N\exp\left[Mx^2\right]
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
N &= \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \left\{\frac{\tilde{C}}{\left(\alpha - 2\tilde{B}\right)}\right\}^{\frac{1}{2}}, \\
M &= \tilde{A} + \frac{\tilde{C}^2}{2\left(\alpha - 2\tilde{B}\right)}
\end{aligned} \tag{14}$$

$$\rho_\omega(x) = |\phi_\omega(x)|^2 = N^*N \exp\left[-\left(M^*+M\right)x^2\right] \tag{15}$$

$$P_\omega = \int_{-\infty}^{+\infty} dx \rho(x) = N^*N \sqrt{\frac{\pi}{(M^*+M)}} \tag{16}$$

Under some physical situations (such as the LHC transversal case) we have, $P_\omega \approx \frac{1}{\sqrt{\alpha}} \frac{mb}{\hbar} \frac{\omega}{\sin(\omega T)}$. In the approximate formulae for P and P_ω, when applicable, the parameter τ does not play a significant role.

5 Longitudinal motion

As far as the longitudinal motion is concerned the quantum-like approach appears to be quite appropriate to obtain information on the modified length (and consequently the stability) of the bunches both in the linear and circular accelerators.. To be more specific it describes a large number of important nonlinear phenomena that are present in RF particle accelerators (with residual addition of longitudinal coupling impedance) as well as in cold plasmas [8].

We introduce the Gaussian parameter b, as we did with the Gaussian slit $e^{-x^2/2b^2}$ in the transversal motion and look for a phenomenological solution of the equation for the beam wave function ψ

$$\mathrm{i}\epsilon_N \partial_t \psi = -\frac{\epsilon_N^2}{2\gamma^3 m_0}\partial_x^2\psi + \frac{1}{2}m_0\omega^2x^2\psi + \Lambda|\psi|^2 \tag{17}$$

where ω is the synchrotron frequency, Λ represents the coupling with non-linear terms and x is the longitudinal particle displacement with respect to the synchrotronous one.

The Feynman propagator is given by (11) and the initial wave function can be again assumed as a Gaussian wave packet. The main difference with the transversal case stays in the numerical values of the parameters that exhibit a different physical situation and require a different physical interpretation.

6 Preliminary estimates

Examples of the numerical calculations for two projects (LHC for ions and HIDIF for heavy ions) with very different physical characteristics are reproduced in the following Tables.

Table 1. Circular machines: transversal case

Parameters	LHC (at injection)	HIDIF (storage ring)
Normalized Transverse Emittance	3.75 mm mrad	13.5 mm mrad
Total Energy, E	450 GeV	5 Gev
$\frac{1}{\sqrt{\alpha}}$	1.2 mm	1.0 mm
T	25 nano sec.	100 nano sec.
τ	88 sec.	4.66 sec.
b	1.2 mm	1.0 mm
$\frac{1}{\sqrt{\tilde{\alpha}}}$	1.41×10^9 m	1.96×10^7 m
P	3.39×10^{-5}	2.37×10^{-3}
ω	4.44×10^6 Hz	1.15×10^7 Hz
$\frac{1}{\sqrt{\tilde{\alpha}_\omega}}$	1.03×10^2 m	2.07×10^{-1} m
P_ω	3.40×10^{-5}	3.00×10^{-3}

Table 2. Circular machines: longitudinal case

Parameters	LHC (at injection)
Normalized Longitudinal Emittance	1.00 eV sec.
Total Energy, E	450 GeV
$\frac{1}{\sqrt{\alpha}}$	7.7 cm
T	25 nano sec.
τ	88 sec.
b	7.7 m
ω	4.23×10^2 Hz
$\frac{1}{\sqrt{\tilde{\alpha}_\omega}}$	1.14×10^6 m
P_ω	0.575

Table 3. RF main LINAC of HIDIF

Parameters	
Normalized Longitudinal Emittance	0.7 keV nano sec.
Total Final Energy, E	5 Gev
$\frac{1}{\sqrt{\alpha}}$	15 cm
T	75 micro sec.
τ	4.9×10^{-4} sec.
b	15 m
ω	4.13×10^5 Hz
$\frac{1}{\sqrt{\tilde{\alpha}_\omega}}$	6.72×10^{-2} m
P_ω	0.707

The machine parameters of Tables 1, 2 and 3 are derived from [6], [7]. In particular ω of Table 3 is calculated on the basis of the "Main LINAC" Table (page 198 of [7]) with the standard formula:

$$\omega^2 = -\frac{eE\omega_{RF}\sin(\phi_s)}{m\beta^3c^3} \tag{18}$$

where the symbols have the usual meaning.

7 Comments and conclusions

Transversal Motion: This use of a quantum-like approach appears a simple powerful tool for the analysis of the evolution of a beam in linear and circular accelerators and storage rings.

Indeed the introduction of a very limited number of phenomenological parameters (in our simplified model the only parameter b) in the beam quantum-like equations and the use of the Schrödinger-type solutions allow us to calculate how the bunches evolve and modify owing to the forces (linear and non-linear) acting on the particles.

As far as the betatronic oscillations are concerned the mechanism of the diffraction through a slit appears a very adequate phenomenological approach. Indeed we can interpret the probability (local and total) for a particle leaving its position as the mechanism of creating a **halo** around the main flux.

The values of τ, ω are strictly connected with the characteristic parameters of the designs of the accelerators (in our example LHC and HIDIF)

The phenomenological parameter b represents several fundamental processes that are present in the beam bunches (and play a determinant role in the creation of the halo) such as intrabeam scattering, beamstrahlung, space-charge and imperfections in the magnets of the lattice that could cause non-linear perturbative effects.

We like to recall here the analogy with the diffraction through a slit in optics where it represents a much more complicated physical phenomenon based on the scattering of light against atomic electrons.

τ is the total time spent in the accelerator by a single bunch, T may coincide with the average time interval between two successive injections and ω is the betatronic average frequency given by $2\pi Q f_r$, f_r being the revolution frequency.

The fact that a small number of parameters can take into account many physical processes is a very nice feature of the quantum-like diffraction approach. However the deep connection between this method and the actual physical process as well as the nonlinear dynamical classical theory is necessary to be understood.

We remark now the following points

1. The total probability (per particle) calculated from the free particle propagator (P) and from the harmonic oscillator one (P_ω) appear very near for the two different circular systems, LHC and HIDIF.
2. The local distribution between the two however looks quite different for the free and harmonic oscillator case, thus giving us a profile of the halo which appears particularly interesting in the HIDIF case (final Gaussian width $\sim \frac{1}{\sqrt{\tilde{\alpha}}} \sim 2.07 \times 10^{-1}$ m)
3. The HIDIF scenario, as we expect because of the higher intensity, exhibits a total loss of particles (and beam power) which is at least 10^3 times higher than LHC. The picture we have obtained for the transversal motion in the two analyzed examples (on the basis of the parameters provided by the latest designs) is encouraging because the halo losses are under control. In both cases the estimated losses of the beam power appear much smaller than the permissible 1 Watt/m.

Longitudinal motion The formulae (7) and (13) can be used for calculating the motion of the length of the bunch related to the synchrotron oscillations in both linear and circular machines. In this case we must consider only the propagator of the harmonic oscillator which is the simplest linear version of the classical dynamical motion for the two canonical conjugate variables that express the deviations of an arbitrary particle from the synchronous one namely the RF phase difference $\Delta\phi = \phi - \phi_s$ and the energy difference $\Delta E = E - E_s$. Our examples are again the LHC synchrotron oscillations and the ones of the main LINAC in the HIDIF project. The phenomenological Gaussian function $e^{-x^2/2b^2}$ acquires a different meaning from the one it had in the transversal motion. Our analysis deals with a Gaussian longitudinal profile and predicts a coasting beam in LHC and a quite stable bunch in the main LINAC of HIDIF.

We may therefore conclude that our approach although preliminary is interesting and particular attention is required in treating the longitudinal motion where the nonlinear space-charge forces are very important. So the quantum-like method appears promising for the future simulations in beam physics.

References

1. See R. Fedele and G. Miele, *Il Nuovo Cimento* D **13**, 1527 (1991); R. Fedele, F. Gallucio, V. I. Man'ko and G. Miele, *Phys. Lett.* A **209**, 263 (1995); Ed. R. Fedele and P.K. Shukla *Quantum-Like Models and Coherent Effects*, Proc. of the 27th Workshop of the INFN Eloisatron Project Erice, Italy 13-20 June 1994 (World Scientific, 1995); R. Fedele, "Quantum-like aspects of particle beam dynamics", *in*: *Proceedings of the 15th Advanced ICFA Beam Dynamics Workshop on Quantum Aspects of beam Physics*, *Ed.* P. Chen, (World Scientific, Singapore, 1999).
   See also: N. C. Petroni, S. De Martino, S. De Siena, and F. Illuminati, A stochastic model for the semiclassical collective dynamics of charged beams in particle accelerators, *in*: *Proceedings of the 15th Advanced ICFA Beam Dynamics Workshop on Quantum Aspects of beam Physics*, *Ed.* P. Chen, (World Scientific, Singapore, 1999)
2. Sameen A. Khan and Modesto Pusterla, **Quantum mechanical aspects of the halo puzzle**, *in*: Proceedings of the *1999 Particle Accelerator Conference* **PAC99** (29 March - 02 April 1999, New York City, NY) *Editors* A. Luccio and W. MacKay, (IEEE Catalogue Number: 99CH36366) pp. 3280-3281.
   Sameen A. Khan and Modesto Pusterla, **Quantum-like approaches to the beam halo problem**, *To appear in*: Proceedings of the *6th International Conference on Squeezed States and Uncertainty Relations* **ICSSUR'99**, (24 - 29 May 1999, Napoli, Italy) (NASA Conference Publication Series)
3. E. Nelson, Phys. Rev. **50** 1079 (1966); Dynamical theories of Brownian motion (Princeton University Press, Princeton 1967)
4. Francesco Guerra, Phys. Rep. **77** 263-312 (1981)
5. Formulae (3-33) *in* R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, (McGraw-Hill, New York)
6. *Ed.* P. Lefèvre and T. Pettersson, Large Hadron Collider (LHC) Conceptual Design CERN/AC/95-05(LHC) (October 1995)
7. *Ed.* I. Hofmann and G. Plass, Heavy Ion Driven Inertial Fusion (HIDIF) Study GSI-98-06 Report (August 1998)
8. R. Fedele and V. G. Vaccaro, Physica Scripta **T52** 36-39 (1994)